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On the Kashiwara–Vergne conjecture

A. Alekseev[1], **E. Meinrenken**[2]

[1] University of Geneva, Section of Mathematics, 2–4 rue du Lièvre, c.p. 64, 1211 Genève 4, Switzerland (e-mail: alekseev@math.unige.ch)
[2] University of Toronto, Department of Mathematics, 40 St George Street, Toronto, Ontario M5S 2E4, Canada (e-mail: mein@math.toronto.edu)

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Abstract. Let G be a connected Lie group, with Lie algebra $\mathfrak{g}$. In 1977, Duflo constructed a homomorphism of $\mathfrak{g}$-modules $\mathrm{Duf}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$, which restricts to an algebra isomorphism on invariants. Kashiwara and Vergne (1978) proposed a conjecture on the Campbell-Hausdorff series, which (among other things) extends the Duflo theorem to germs of bi-invariant distributions on the Lie group G.

The main results of the present paper are as follows. (1) Using a recent result of Torossian (2002), we establish the Kashiwara–Vergne conjecture for any Lie group G. (2) We give a reformulation of the Kashiwara–Vergne property in terms of Lie algebra cohomology. As a direct corollary, one obtains the algebra isomorphism $H(\mathfrak{g}, S(\mathfrak{g})) \to H(\mathfrak{g}, U(\mathfrak{g}))$, as well as a more general statement for distributions.

1. Introduction

Let G be a connected Lie group, with Lie algebra $\mathfrak{g}$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra, and $S(\mathfrak{g})$ the symmetric algebra. The Duflo map is the isomorphism of $\mathfrak{g}$-modules,

$$\mathrm{Duf}\colon S(\mathfrak{g}) \to U(\mathfrak{g}), \tag{1}$$

obtained by precomposing the symmetrization map with an infinite-order differential operator $\widehat{J^{1/2}}$ on the symmetric algebra (viewed as the space of polynomials on $\mathfrak{g}^*$). Here $J \in C^\infty(\mathfrak{g})$ is the analytic function

$$J(x) = \det_{\mathfrak{g}}\Big(\frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\Big), \tag{2}$$

and $\widehat{J^{1/2}}\colon S(\mathfrak{g}) \to S(\mathfrak{g})$ is obtained from the Taylor series expansion of $J^{1/2}$ by replacing the variables x with derivatives $\frac{\partial}{\partial\mu}$. Duflo's theorem [6] states that the map Duf restricts to an *algebra* isomorphism on invariants, $S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$. In [8] Kashiwara and Vergne proposed a conjecture regarding the Campbell-Hausdorff series, which among other things generalizes the Duflo theorem to germs of invariant distributions on the Lie group G. Here $S(\mathfrak{g}) \subset \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g})$ is identified with the subalgebra (under convolution) of distributions on $\mathfrak{g}$ supported at 0, while $U(\mathfrak{g})$ is identified with the subalgebra (under convolution) of distributions on G supported at e. This generalization of Duflo's theorem was proved later in a series of papers [3–5] using the diagrammatic technique of Kontsevich [9]. The KV conjecture itself was established for $\mathfrak{g}$ solvable in [8], for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ in [12], and in the case of $\mathfrak{g}$ quadratic (that is, $\mathfrak{g}$ carries an invariant nondegenerate symmetric bilinear form) in [15] (see also [1]).

There are several equivalent formulations of the KV conjecture, one of which is as follows (see Sect. 2 for details). Let $\mathfrak{g}_t$ denote the family of Lie algebras, obtained from $\mathfrak{g}_1 = \mathfrak{g}$ by rescaling the Lie bracket as $[\cdot,\cdot]_t = t[\cdot,\cdot]$, and let m_t denote the family of products on $S(\mathfrak{g})$, induced from the products on $U(\mathfrak{g}_t)$ by the Duflo isomorphisms $\mathrm{Duf}_t\colon S(\mathfrak{g}) \to U(\mathfrak{g}_t)$. Thus, m_t interpolates between the usual commutative product m_0 on $S(\mathfrak{g})$ and the non-commutative product coming from $U(\mathfrak{g})$. The product maps m_t have a natural extension to the space of distributions on $\mathfrak{g}$, with compact support in a sufficiently small open neighborhood O of 0,

$$m_t : \mathcal{D}'_{\mathrm{comp}}(O^2) \to \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g}). \tag{3}$$

Let e_i be a basis of $\mathfrak{g}^2 = \mathfrak{g} \oplus \mathfrak{g}$, and let $L(e_i)$ be the Lie derivatives relative to the adjoint action of $\mathfrak{g}^2$ on $\mathcal{D}'_{\mathrm{comp}}(O^2)$. The Kashiwara–Vergne conjecture asserts the existence of an analytic map $\beta = \sum \beta^i e_i : O^2 \to \mathfrak{g}^2$, vanishing at the origin, such that

$$\frac{dm_t}{dt} = -m_t \circ \sum_i \beta_t^i L(e_i), \tag{4}$$

where $\beta_t^i(x,y) = t^{-1}\beta^i(tx,ty)$. Equation (4) implies that the family of products m_t is independent of t when restricted to germs of invariant distributions.

In a recent paper, Torossian [14] came very close to a proof of the KV conjecture, again using Kontsevich diagram techniques. In more detail, Torossian proved an analogue of Equation (4), for a certain family of maps $\tilde{m}_t : \mathcal{D}'_{\mathrm{comp}}(O^2) \to \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g})$ with $\tilde{m}_0 = m_0$, $\tilde{m}_1 = m_1$. However, $\tilde{m}_t \neq m_t$, and indeed the products defined by $\tilde{m}_t$ are not associative.

Our first main result shows that one obtains a solution β_t of the KV equation (4) from a solution $\tilde{\beta}_t$ of Torossian's modified KV property, by solving an interesting 'zero curvature' equation. This establishes the Kashiwara–Vergne conjecture in full generality.

Our second result relates the KV property to Lie algebra cohomology, as follows. For any $\mathfrak{g}$-module X, let $H^\bullet(\mathfrak{g}, X)$ denote the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in X. By functoriality, the Duflo map (1) induces a homomorphism of graded vector spaces,

$$H^\bullet(\mathfrak{g}, S(\mathfrak{g})) \to H^\bullet(\mathfrak{g}, U(\mathfrak{g})). \tag{5}$$

In [13], Shoikhet announced a joint result with Kontsevich that the linear map (5) is an isomorphism of graded algebras. Shortly later, a proof was published by Pevzner-Torossian [11]. In his paper, Shoikhet raises the question [13, Remark 3.2] whether this result would follow from the KV conjecture. We show that this is indeed the case: Let

$$M_t\colon \mathcal{D}'_{\mathrm{comp}}(O^2) \otimes \wedge(\mathfrak{g}^2)^* \to \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g}) \otimes \wedge\mathfrak{g}^* \tag{6}$$

denote the family of products on Chevalley-Eilenberg complexes, induced by the maps m_t (3). We prove that $\beta : O^2 \to \mathfrak{g}^2$ solves the KV conjecture if and only if

$$\frac{dM_t}{dt} = -\Big[\mathrm{d}, M_t \circ \sum_i \beta_t^i \otimes \iota(e_i)\Big]. \tag{7}$$

Here d is the Chevalley-Eilenberg differential, and $\iota(e_i)$ are contraction operators. Equation (7) implies that the product maps M_t are chain homotopic. Hence, the induced map in cohomology is independent of t.

The organization of this paper is as follows. In Sect. 2, we give three different formulations I, II, III of the KV conjecture. The algebraic formulation III is the main version of the conjecture, as stated in [8, p. 250]. The other two versions are more geometric. They are implicit in [8, Sect. 3], although we add some details to show that they are indeed equivalent to III. In Sect. 3 we show that Torossian's theorem [14] implies the KV conjecture. Section 4 gives yet another reformulation IV of the KV conjecture, as a homotopy formula for Chevalley-Eilenberg complexes. In this context, we formulate a new conjecture extending the Kashiwara–Vergne approach to higher homotopies.

Notational conventions. For any manifold M, we denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields (viewed as derivations of $C^\infty(M)$), and by $\Omega^\bullet(M)$ the algebra of differential forms. We follow the convention that the flow F_t of a time dependent vector field X_t is given by $(X_t f)(F_t^{-1}(x)) = \frac{\partial}{\partial t} f(F_t^{-1}(x))$. The Lie derivative on $\Omega(M)$ is thus given by $F_t^* \circ L_{X_t} = -\frac{\partial}{\partial t} F_t^*$. A family of forms α_t is intertwined by the flow if $F_t^* \alpha_t$ is independent of t; in terms of the Lie derivative this means $(\frac{\partial}{\partial t} - L(X_t))\alpha_t = 0$.

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2. Review of the Kashiwara–Vergne conjecture

2.1. Preliminaries. Throughout this paper, r_t^V will denote scalar multiplication by a real parameter t on some vector space V. We omit the superscript if V is clear from the context. We will frequently use the following elementary fact. Suppose $R_t\colon W \to W,\ 0 < t \le 1$ are automorphisms of a topological vector space, with $R_{t_1 t_2} = R_{t_1} \circ R_{t_2}$. Assume that $w \in W$ be a smooth vector, i.e. $w_t = R_t(w)$ is differentiable in t. Then the derivative $\dot{w}_t = \frac{\partial w_t}{\partial t}$ satisfies

$$\dot{w}_t = t^{-1} R_t(\dot{w}_1). \tag{8}$$

For example, suppose O is a star-shaped open subset of a vector space V, and take W to be the space of k-forms on O. Let $\alpha \in \Omega^k(O)$ and $a \in \mathbb{R}$. Then the derivative of $\alpha_t := t^a r_t^* \alpha$ scales according to $\dot{\alpha}_t = t^{a-1} r_t^* \dot{\alpha}_1$.

2.2. The Duflo map on distributions. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{R}$, and G the corresponding connected, simply connected Lie group. Let $\exp\colon \mathfrak{g} \to G$ denote the exponential map. The Jacobian $J \in C^\infty(\mathfrak{g})$ of the exponential map (using left trivialization of the tangent bundle of G) is given by formula (2). Let $\mathcal{D}'_{\mathrm{comp}}(\mathfrak{g})$ denote the commutative algebra of compactly supported distributions on $\mathfrak{g}$, with product given by convolution. The symmetric algebra $S(\mathfrak{g})$ is identified with the subalgebra of distributions supported at 0. Similarly, let $\mathcal{D}'_{\mathrm{comp}}(G)$ denote the non-commutative convolutions algebra of distributions on G, and identify $U(\mathfrak{g})$ with distributions supported at the group unit $e \in G$. Assume temporarily that J admits a global square root $J^{1/2}$, equal to 1 at the origin. (This is not the case for all G, but holds true, for instance, if $\mathfrak{g}$ admits an invariant non-degenerate symmetric bilinear form.) Then the Duflo map $\mathrm{Duf}\colon S\mathfrak{g} \to U\mathfrak{g}$ extends to a homomorphism of $\mathfrak{g}$-modules,

$$\mathrm{Duf} = \exp_* \circ J^{1/2}\colon \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g}) \to \mathcal{D}'_{\mathrm{comp}}(G), \tag{9}$$

and again this is an algebra homomorphism on invariants. For a general non-compact Lie group, this extension may not be very interesting since there may be very few invariant distributions of compact support. However, one obtains an interesting generalization by working instead with *germs* of invariant distributions. Let $\mathfrak{D}'(\mathfrak{g}, A)$ denote the space of germs (at the origin $0 \in \mathfrak{g}$) of distributions on $\mathfrak{g}$, with asymptotic support[1] in the closed cone $A \subset \mathfrak{g}$. Let $\mathfrak{D}'(G, A)$ be the space of germs of distributions at $e \in G$, given as the image of $\mathcal{D}'(\mathfrak{g}, A)$ under $\exp_*$. For closed cones $A_1, A_2 \subset \mathfrak{g}$ with $A_1 \cap -A_2 = \{0\}$, there are well-defined convolution products

$$\mathfrak{D}'(\mathfrak{g}, A_1) \otimes \mathfrak{D}'(\mathfrak{g}, A_2) \to \mathfrak{D}'(\mathfrak{g}, A_1 + A_2), \tag{10}$$

[1] Let x be a manifold, and $\mathfrak{u}$ the germ of a distribution at $x \in M$. The *(asymptotic) support* of $\mathfrak{u}$ is the closed subset $\mathrm{supp}_x(\mathfrak{u}) \subset T_x M$, defined as follows: Let $\gamma(t)$ denote a smooth curve with $\gamma(0) = x$. Then the tangent vector $\dot{\gamma}(0) \notin \mathrm{supp}_x(\mathfrak{u})$ if and only if $\gamma(t) \notin \mathrm{supp}(\mathfrak{u})$ for $t > 0$ sufficiently small.

and

$$\mathfrak{D}'(G, A_1) \otimes \mathfrak{D}'(G, A_2) \to \mathfrak{D}'(G, A_1 + A_2). \tag{11}$$

For instance, these products are well-defined if $A_1 = \{0\}$ or $A_2 = \{0\}$. The natural extension of Duflo's theorem asserts that the Duflo map intertwines (10) (restricted to $\mathfrak{g}$-invariants), with (11) (restricted to $\mathfrak{g}$-invariants). This extension of Duflo's theorem was put forward by Kashiwara–Vergne [8] in their paper, and proved for the case of solvable Lie algebras. The general case was proved many years later in [3–5].

2.3. The Kashiwara–Vergne approach. The Kashiwara–Vergne approach is based on the idea that the Duflo theorem (and its extension to germs of distributions) should follow from a more general property of the Campbell-Hausdorff series

$$\Phi(x, y) := \log(\exp(x)\exp(y)) = x + y + \tfrac{1}{2}[x, y] + \cdots . \tag{12}$$

Recall that the function $\Phi(x, y)$ is well-defined and analytic for x, y in a sufficiently small neighborhood of the origin. To be specific, let $U \subset \mathfrak{g}$ be a star-shaped neighborhood of the origin such that the exponential map is a diffeomorphism over U, and let $O \subset U$ be a smaller star-shaped open neighborhood such that $\exp(x)\exp(y) \in \exp(U)$ for all $x, y \in O$. Then $\Phi\colon O^2 \to \mathfrak{g}$ is well-defined and takes values in U. Since $J > 0$ on U, the square root $J^{1/2}$ is a well-defined analytic function on U. Under the Duflo map, the convolution product on $\mathcal{D}'_{\mathrm{comp}}(G)$ gives rise to a non-commutative product on distributions on $\mathfrak{g}$ with support in O. In terms of the map Φ this product can be written,

$$m = \Phi_* \circ \kappa\colon \mathcal{D}'_{\mathrm{comp}}(O^2) \to \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g}) \tag{13}$$

where $\kappa \in C^\infty(O^2)$ is the function,

$$\kappa(x, y) = \frac{J^{1/2}(x) J^{1/2}(y)}{J^{1/2}(\Phi(x, y))}. \tag{14}$$

Denote by $\mathfrak{g}_t$ the Lie algebra, obtained from $\mathfrak{g}$ by rescaling the Lie bracket,

$$[\cdot, \cdot]_t = t[\cdot, \cdot]. \tag{15}$$

Let G_t denote the connected, simply connected Lie group with Lie algebra $\mathfrak{g}_t$. For $t \neq 0$, the map $r_t\colon \mathfrak{g} \to \mathfrak{g}$ defines a Lie algebra isomorphism $\mathfrak{g}_t \to \mathfrak{g}_1$, while $\mathfrak{g}_0$ is the vector space $\mathfrak{g}$ with the zero bracket. The exponential map $\exp_t\colon \mathfrak{g}_t \to G_t$ is a diffeomorphism over $U_t = r_{t^{-1}} U$, and the counterpart $\Phi_t\colon O_t \to \mathfrak{g}$ of the map Φ is well-defined over the rescaled neighborhood $O_t = r_{t^{-1}} O$. One has $\Phi_t = t^{-1} r_t^* \Phi$ for $t \neq 0$, i.e.

$$\Phi_t(x, y) = x + y + \tfrac{t}{2}[x, y] + \cdots , \tag{16}$$

while $\Phi_0(x, y) = x + y$. The Jacobian of $\exp_t$ is $J_t = r_t^* J$. From the family of Lie brackets $[\cdot, \cdot]_t$, we obtain a family of multiplication maps $m_t = (\Phi_t)_* \circ \kappa_t$ with $\kappa_t = r_t^* \kappa$. For $t \neq 0$,

$$m_t = (r_{t^{-1}})_* \circ m_1 \circ (r_t)_*. \tag{17}$$

The map m_0 is the commutative product given by convolution on the vector space $\mathfrak{g}$. The t-derivative $\dot{m}_t = \frac{\partial m_t}{\partial t}$ scales according to

$$\dot{m}_t = t^{-1} (r_{t^{-1}})_* \circ \dot{m}_1 \circ (r_t)_*. \tag{18}$$

The generalized Duflo theorem is equivalent to the statement that the derivative $\dot{m}_t$ vanishes on $\mathcal{D}'(\mathfrak{g} \times \mathfrak{g}, A_1 \times A_2)^{\mathfrak{g}\times\mathfrak{g}}$, where $A_1 \cap (-A_2) = \{0\}$. Kashiwara–Vergne formulated the following stronger property of m_t.

2.4. The Kashiwara–Vergne conjecture. Let e_i be a basis of $\mathfrak{g}^2$, and let $L_i = L(e_i)$ be the Lie derivatives for the $\mathfrak{g}^2$-action on $\mathcal{D}'_{\mathrm{comp}}(O^2)$.

Kashiwara–Vergne conjecture, version I: *Taking O smaller if necessary, there exists a $\mathfrak{g}$-equivariant smooth function $\beta = \sum_i \beta^i e_i \colon O^2 \to \mathfrak{g}^2$, with $\beta(0,0) = 0$, such that $\beta_t = t^{-1} r_t^* \beta$ satisfies,*

$$\frac{\partial m_t}{\partial t} = -m_t \circ \sum_i \beta_t^i L_i. \tag{19}$$

Duflo's theorem, as well as its extension to germs of invariant distributions, are a direct consequence of (19) since the right hand side vanishes on invariants.

Remark 2.1. It is not difficult to show that $\dot{m}_t$ can be written in the form

$$\dot{m}_t = -m_t \circ P_t \tag{20}$$

where P_t are first order differential operators on $\mathcal{D}'_{\mathrm{comp}}(O^2)$. In light of Duflo's theorem, the idea that one can take P_t of the form $\sum_i \beta_t^i L_i$ may therefore appear rather natural. Note also that the operator $\sum_i \beta_t^i L_i$ on $\mathcal{D}'(O^2)$ is different, in general, from the Lie derivative $L(X^{\beta_t})$ in the direction of the vector field X^{β_t} generated by β_t, since on distributions $L(X^{\beta_t}) = \sum_i L_i \circ \beta_t^i$.

Another version of the conjecture states the KV property in terms of the derivatives of the function Φ_t and κ_t.

Kashiwara–Vergne conjecture, version II: *Taking O smaller if necessary, there exists a $\mathfrak{g}$-equivariant smooth function $\beta \colon O^2 \to \mathfrak{g}^2$, with $\beta(0,0) = 0$,* *such that* $\beta_t = t^{-1} r_t^* \beta$ *satisfies the following two equations:*

$$\frac{\partial \Phi_t}{\partial t} - L(X^{\beta_t})\Phi_t = 0, \tag{21}$$

$$\frac{\partial \kappa_t}{\partial t} - \Big(\sum_i L_i \circ \beta_t^i\Big)\kappa_t = 0. \tag{22}$$

Remark 2.2. The geometric interpretation of the first equation is that the vector field X^{β_t} intertwines the product maps, Φ_t. To interpret the second equation, choose a translation invariant volume form Γ_0 on $\mathfrak{g}^2$, and consider the family of volume forms $\Gamma_t = \kappa_t \Gamma_0$. Then the second equation is equivalent to

$$\begin{aligned}\dot{\Gamma}_t &= L(X^{\beta_t})\Gamma_t - \sum_i \beta_t^i \kappa_t L_i \Gamma_0 \\ &= L(X^{\beta_t})\Gamma_t - \sum_i \beta_t^i \operatorname{tr}_{\mathfrak{g}\times\mathfrak{g}}(e_i)\Gamma_t\end{aligned}$$

where we have used that $L(X^{\beta_t}) = \sum_i L_i \circ \beta_t^i$ on volume forms. If $\mathfrak{g}$ is *unimodular* (i.e., $\operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(x)) = 0$ for all x), the equation says that the vector field X^{β_t} intertwines the volume forms Γ_t, while in the general case Γ_t is a relative invariant for the modular character.

The third version of the Kashiwara–Vergne conjecture is more algebraic, and no longer involves the variable t. This is the 'main' version of the conjecture, as formulated in [8].

Kashiwara–Vergne conjecture, version III: *Taking* O *smaller if necessary, there exist smooth* $\mathfrak{g}$*-equivariant functions* $\beta^1, \beta^2 \colon O^2 \to \mathfrak{g}$*, vanishing at* $(0,0)$*, such that the following two equations are satisfied:*

$$(1 - e^{-\operatorname{ad}_x})\beta^1(x,y) + (e^{\operatorname{ad}_y} - 1)\beta^2(x,y) = x + y - \log(\exp(y)\exp(x)), \tag{23}$$

$$\begin{aligned}&-\operatorname{tr}\Big(\operatorname{ad}_x \circ \partial_x \beta^1(x,y) + \operatorname{ad}_y \circ \partial_y \beta^2(x,y)\Big) \\ &= \tfrac{1}{2}\operatorname{tr}\Big(\frac{\operatorname{ad}_x}{e^{\operatorname{ad}_x} - 1} + \frac{\operatorname{ad}_y}{e^{\operatorname{ad}_y} - 1} - \frac{\operatorname{ad}_z}{e^{\operatorname{ad}_z} - 1} - 1\Big),\end{aligned} \tag{24}$$

with $z = \log(\exp(x)\exp(y))$*. Here* tr *denotes the trace in the adjoint representation,* $\partial_x \beta^1(x,y) \in \operatorname{End}(\mathfrak{g})$ *is the linear map*

$$\xi \mapsto \frac{\partial}{\partial u}\Big|_{u=0} \beta^1(x + u\xi, y), \tag{25}$$

and similarly for $\partial_y \beta^2(x,y)$.

Remark 2.3. A *Lie polynomial* in variables $x_1, \dots, x_l$ is an element of the free Lie algebra $\mathfrak{l} = \mathfrak{l}(x_1, \dots, x_l)$ with generators x_i. Recall that the free Lie algebra has a grading $\mathfrak{l} = \bigoplus_k \mathfrak{l}^k$, where the degree k component consists of linear combinations of *Lie monomials* $[z_{i_1}, [z_{i_2}, \dots, [z_{i_{k-1}}, z_{i_k}], \dots]]$ where $z_j \in \{x_1, \dots, x_l\}$. A *Lie series* is an element of the degree completion $\widehat{\mathfrak{l}} = \prod_k \mathfrak{l}^k$. If x_i are elements of a given Lie algebra $\mathfrak{g}$, one obtains a Lie algebra homomorphism $\mathfrak{l} \to \mathfrak{g}$ taking $x_i \in \mathfrak{l}$ to $x_i \in \mathfrak{g}$.

The Campbell-Hausdorff theorem (see e.g. [7, Chapt. I.3]) states that if $\dim \mathfrak{g} < \infty$, the function $\Phi(x, y) = \log(\exp(x)\exp(y))$ is a Lie series in x, y. Dynkin's formula gives this series explicitly as a *universal* element in $\widehat{\mathfrak{l}}(x, y)$ (i.e. not depending on the Lie algebra). It is hence natural to require that similarly, β^1, β^2 are analytic functions in x, y given as universal Lie series.

2.5. Equivalence of versions I–III of the KV conjecture. The implication $III \Rightarrow II \Rightarrow I$ was shown in [8, Sect. 3]. Closer examination of their argument shows that the three statements are in fact equivalent:

2.5.1. Equivalence $I \Leftrightarrow II$. We have to show that $\dot{m}_t = -m_t \circ \sum_i \beta_t^i L_i$ if and only if β_t solves the pair of equations (21), (22). In fact, this equivalence does not involve the particular form of Φ_t, κ_t. Thus let $O \subset \mathfrak{g}$ be an open neighborhood of the origin, and let $\Phi_t : O^2 \to \mathfrak{g}$, $\kappa_t : O^2 \to \mathbb{R}$ be *arbitrary* families of smooth functions. Define a family of maps $\mathcal{D}'_{\mathrm{comp}}(O^2) \to \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g})$ by $m_t = (\Phi_t)_* \circ \kappa_t$. Let $\beta_t : O^2 \to \mathfrak{g}^2$ be a smooth family of functions.

Lemma 2.4. *The equation $\dot{m}_t = -m_t \circ \sum_i \beta_t^i L_i$ holds if and only if*

$$\dot{\Phi}_t - L(X^{\beta_t})\Phi_t = 0, \quad \dot{\kappa}_t - \Big(\sum_i L_i \circ \beta_t^i\Big)\kappa_t = 0. \tag{26}$$

Proof. Recall that linear combinations of delta-distributions are dense in the space of distributions. Hence $\dot{m}_t = -m_t \circ \sum_i \beta^i L_i$ holds if and only if it holds on $\delta_{(q,p)}$, for all $(q, p) \in O^2$. Let us work out the resulting identities. Taking the t-derivative of

$$m_t(\delta_{(q,p)}) = \kappa_t(q, p)\delta_{\Phi_t(q,p)}, \tag{27}$$

we obtain

$$\dot{m}_t(\delta_{(q,p)}) = \dot{\kappa}_t(q, p)\delta_{\Phi_t(q,p)} - \kappa_t(q, p) \sum_a \dot{\Phi}_t^a(q, p) \frac{\partial}{\partial x^a} \delta_{\Phi_t(q,p)}. \tag{28}$$

Here e_a is a basis of $\mathfrak{g}$ and x^a are the corresponding coordinates. On the other hand,

$$
\begin{aligned}
&\Big(m_t \circ \sum_i \beta_t^i L_i\Big)(\delta_{(q,p)})\\
&\quad = (\Phi_t)_*\Big(\sum_i \kappa_t \beta_t^i L_i\, \delta_{(q,p)}\Big)\\
&\quad = (\Phi_t)_*\Big(-\sum_i \big(L_i\big(\kappa_t\beta_t^i\big)\big)\delta_{(q,p)} + \sum_i \big(\kappa_t\beta_t^i\big)(q,p) L_i\, \delta_{(q,p)}\Big)\\
&\quad = -\sum_i L_i\big(\kappa_t\beta_t^i\big)(q,p)\delta_{\Phi_t(q,p)} + \sum_i \big(\kappa_t\beta_t^i\big)(q,p)(\Phi_t)_*(L_i\, \delta_{(q,p)})\\
&\quad = -\sum_i L_i\big(\kappa_t\beta_t^i\big)(q,p)\delta_{\Phi_t(q,p)} + \sum_{ia} \big(\kappa_t\beta_t^i\, L_i\big(\Phi_t^a\big)\big)(q,p)\frac{\partial}{\partial x^a}\delta_{\Phi_t(q,p)}.
\end{aligned}
$$

Comparing the coefficients of $\delta_{\Phi(p,q)}$, $\frac{\partial}{\partial x^a}\delta_{\Phi_t(q,p)}$ with those in Equation (28), we find that $\dot{m}_t = -m_t \circ \sum_i \beta^i L_i$ is equivalent to

$$
\dot{\Phi}_t^a = \sum_i \beta_t^i L_i\big(\Phi_t^a\big), \quad \dot{\kappa}_t = \sum_i L_i\big(\kappa_t\beta_t^i\big), \tag{29}
$$

which is (26). □

2.5.2. Equivalence II $\Leftrightarrow$ III. Let $A(x,y)$, $A'(x,y)$ denote the left-, right-hand side of (23), and write $A_t = t^{-1}r_t^* A$ and $A_t' = t^{-1}r_t^* A'$. In the proof of [8, Lemma 2.3], Kashiwara–Vergne compute

$$
\dot{\Phi}_t(x,y) = \frac{1}{t}\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \log\big(e^{tx}e^{\epsilon A_t'(x,y)}e^{ty}\big)
$$

$$
-\big(L(X^{\beta_t})\Phi_t\big)(x,y) = \frac{1}{t}\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \log\big(e^{tx}e^{\epsilon A_t(x,y)}e^{ty}\big).
$$

It follows that the first condition (21) of version II is equivalent to $A_t = A_t'$, hence to $A = A'$ which is the first condition (23) of version III.

Assuming (21), we now show that (22) is equivalent to (24). Let $B(x,y)$, $B'(x,y)$ denote the left-, right-hand side of Equation (24), and let $B_t = t^{-1}r_t^* B$, $B_t' = t^{-1}r_t^* B'$. Then

$$
B_t = -\sum_i \big(L_i\beta_t^i\big), \tag{30}
$$

and Equation (22) may be written in the form

$$
0 = \dot{\kappa}_t - L(X^{\beta_t})\kappa_t + \sum_i (L_i\beta_t^i)\kappa_t = \Big(\frac{\partial}{\partial t} - L(X^{\beta_t})\Big)\kappa_t - B_t\kappa_t. \tag{31}
$$

From (21), we have

$$\Big(\frac{\partial}{\partial t} - L(X^{\beta_t})\Big) J_t(\Phi_t(x, y)) = \dot{J}_t(\Phi_t(x, y)).$$

On the other hand, $L(X^{\beta_t})J_t(x) = L(X^{\beta_t})J_t(y) = 0$ since J is ad-invariant. Recalling the formula [8, Lemma 3.3]

$$J_t(z)^{-1}\dot{J}_t(z) = \operatorname{tr}_{\mathfrak{g}}\Big(\frac{t\,\mathrm{ad}_z}{e^{t\,\mathrm{ad}_z} - 1} - 1\Big), \quad z \in \mathfrak{g} \tag{32}$$

this gives,

$$\dot{\kappa}_t - L(X^{\beta_t})\kappa_t = B'_t\kappa_t. \tag{33}$$

We thus obtain $B_t = B'_t$, i.e. $B = B'$.

3. Torossian's theorem implies the KV conjecture

3.1. Torossian's theorem. In his paper, Torossian [14, p. 601] proves the following statement, similar to version II of the KV conjecture.

Theorem 3.1 (Torossian). *There are families of smooth functions*

$$\tilde{\Phi}_s : O^2 \to \mathfrak{g},\ \tilde{\kappa}_s : O^2 \to \mathbb{R},\ \tilde{\beta}_s : O^2 \to \mathfrak{g}^2, \quad s \in [0, 1] \tag{34}$$

with the following properties:

(a) The Taylor expansion of $\tilde{\Phi}_s(x, y)$ at the origin $(0, 0)$ is of the form $\tilde{\Phi}_s(x, y) = x + y + \cdots$. Furthermore, $\tilde{\kappa}_s(0, 0) = 1$, $\tilde{\beta}_s(0, 0) = 0$,

(b) For $s = 0$,

$$\tilde{\Phi}_0(x, y) = x + y, \quad \tilde{\kappa}_0(x, y) = 1,$$

while for $s = 1$,

$$\tilde{\Phi}_1(x, y) = \Phi(x, y), \quad \tilde{\kappa}_1(x, y) = \kappa(x, y).$$

(c) The following two equations are satisfied:

$$\dot{\tilde{\Phi}}_s - L(X^{\tilde{\beta}_s})\tilde{\Phi}_s = 0, \quad \dot{\tilde{\kappa}}_s - \Big(\sum_i L_i \circ \tilde{\beta}^i_s\Big)\tilde{\kappa}_s = 0.$$

The functions $\tilde{\Phi}_s$, $\tilde{\kappa}_s$, $\tilde{\beta}_s$ are constructed using the diagrammatic technique of Kontsevich [9]. See [14] for details.

Using $\tilde{\Phi}_s$ and $\tilde{\kappa}_s$ one can define maps $\tilde{m}_s = (\tilde{\Phi}_s)_* \circ \tilde{\kappa}_s$ as before. However, $\tilde{m}_s \neq m_s$, and in fact the 'products' $\tilde{m}_s$ are not associative, in general. Nonetheless, according Lemma 2.4 one still has

$$\frac{\partial \tilde{m}_s}{\partial s} = -\tilde{m}_s \circ \sum_i \tilde{\beta}^i_s L_i, \tag{35}$$

which as before implies the Duflo theorem and its generalization to germs of invariant distributions.

3.2. Torossian $\Rightarrow$ KV. We will now show how to obtain a solution β_t of the original KV problem from Torossian's function $\tilde{\beta}_s$. Let us first state the relevant argument in a more general setting. Let $\mathfrak{k}$ be a Lie algebra, acting on a manifold M. The vector fields defining this action are a Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{X}(M),\ \xi \mapsto X^\xi$. For $\beta \in \mathfrak{k}_M := C^\infty(M, \mathfrak{k})$ define $X^\beta \in \mathfrak{X}(M)$ by $X^\beta(x) = X^{\beta(x)}(x)$. Then

$$\mathfrak{k}_M \to \mathfrak{X}(M),\ \beta \mapsto X^\beta \tag{36}$$

is a Lie algebra homomorphism for the following Lie bracket on $\mathfrak{k}_M$,

$$[\beta, \gamma](x) = (L(X^\beta)\gamma)(x) - (L(X^\gamma)\beta)(x) + [\beta(x), \gamma(x)]. \tag{37}$$

The following lemma will be proved in the appendix.

Lemma 3.2. *Suppose $\gamma_{s,t} \in \mathfrak{k}_M$ is a given smooth 2-parameter family, with $\gamma_{s,t}(x_0) = 0$ for a given point $x_0 \in M$. Replacing M by a smaller neighborhood of x_0 if necessary, there exists a unique 2-parameter family $\beta_{s,t} \in \mathfrak{k}_M$ with $\beta_{0,t} = 0$, such that the following zero curvature equation is satisfied:*

$$\frac{\partial \beta_{s,t}}{\partial s} - \frac{\partial \gamma_{s,t}}{\partial t} + [\beta_{s,t}, \gamma_{s,t}] = 0. \tag{38}$$

Let us describe another general feature of the Lie algebra $\mathfrak{k}_M$. For $\beta \in \mathfrak{k}_M$, consider the first order differential operator on $\mathcal{D}'(M)$ (cf. (22))

$$V(\beta) = \sum_i \beta^i \circ L(e_i)\ . \tag{39}$$

The operator $V(\beta)$ is different from the Lie derivative $L(X^\beta)$, which equals $\sum_i L_i \circ \beta^i$ on distributions. However, one verifies that the difference $L(X^\beta) - V(\beta) = \tau(\beta)$, where

$$\tau\colon \mathfrak{k}_M \to C^\infty(M),\ \beta \mapsto \sum_i \big(L_i \beta^i\big), \tag{40}$$

is a Lie algebra cocycle: $\tau([\beta, \gamma]) = L(X^\beta)\tau(\gamma) - L(X^\gamma)\tau(\beta)$. This gives

$$V([\beta, \gamma]) = [V(\beta), V(\gamma)]. \tag{41}$$

Note that for the function β from the Kashiwara–Vergne conjecture (version III), the cocycle $\tau(\beta)$ appears as the left hand side of (24) (see also (30)).

Theorem 3.3. *Any solution $\tilde{\beta}_s$ of Torossian's modified KV problem (cf. Theorem 3.1) determines a solution β of the KV problem.*

Proof. Introduce a 2-parameter family of maps $\tilde{\beta}_{s,t} = r_t^*\tilde{\beta}_s,\quad s,t \in [0,1]$. Since $\tilde{\beta}_s(0,0) = 0$, we have $\tilde{\beta}_{s,t}(0,0) = 0$. Similarly, rescale $\tilde{\Phi}_{s,t} = t^{-1}r_t^*\tilde{\Phi}_s$ and $\tilde{\kappa}_{s,t} = r_t^*\tilde{\kappa}_s$. For each s,t the 'product' $\tilde{m}_{s,t} = (\tilde{\Phi}_{s,t})_* \circ \tilde{\kappa}_{s,t}$ satisfies the equation,

$$\frac{\partial \tilde{m}_{s,t}}{\partial s} = -\tilde{m}_{s,t} \circ \sum_i \tilde{\beta}^i_{s,t} L_i.$$

Now let $\beta_{s,t}$ be the 2-parameter family of maps $\beta_{s,t} : O^2 \to \mathfrak{g}^2,\quad \beta_{0,t} = 0$ obtained by solving the zero curvature equation, Lemma 3.2, with $\gamma_{s,t} = \tilde{\beta}_{s,t}$. The scaling property $\tilde{\beta}_{s,t} = r_t^*\tilde{\beta}_s$ implies the scaling property for the derivative, $\frac{\partial \tilde{\beta}_{s,t}}{\partial t} = t^{-1}r_t^*\big(\frac{\partial \tilde{\beta}_{s,u}}{\partial u}|_{u=1}\big)$, and hence (by Equation (38)) $\beta_{s,t} = t^{-1}r_t^*\beta_{s,1}$. We compute,

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big(\frac{\partial \tilde{m}_{s,t}}{\partial t} + \tilde{m}_{s,t} \circ V(\beta_{s,t})\Big)\\
&= \frac{\partial}{\partial t}\frac{\partial \tilde{m}_{s,t}}{\partial s} + \frac{\partial \tilde{m}_{s,t}}{\partial s} \circ V(\beta_{s,t}) + \tilde{m}_{s,t} \circ V\Big(\frac{\partial \beta_{s,t}}{\partial s}\Big)\\
&= -\frac{\partial}{\partial t}(\tilde{m}_{s,t} \circ V(\tilde{\beta}_{s,t})) - \tilde{m}_{s,t} \circ V(\tilde{\beta}_{s,t}) \circ V(\beta_{s,t}) + \tilde{m}_{s,t} \circ V\Big(\frac{\partial \beta_{s,t}}{\partial s}\Big)\\
&= -\frac{\partial \tilde{m}_{s,t}}{\partial t} \circ V(\tilde{\beta}_{s,t}) - \tilde{m}_{s,t} \circ \Big(V\Big(\frac{\partial \tilde{\beta}_{s,t}}{\partial t}\Big) - V\Big(\frac{\partial \beta_{s,t}}{\partial s}\Big) + V(\tilde{\beta}_{s,t}) \circ V(\beta_{s,t})\Big)\\
&= -\Big(\frac{\partial \tilde{m}_{s,t}}{\partial t} + \tilde{m}_{s,t} \circ V(\beta_{s,t})\Big) \circ V(\tilde{\beta}_{s,t})
\end{aligned}$$

where in the last step we used (38) with $\gamma_{s,t} = \tilde{\beta}_{s,t}$. This equality is an ordinary first order differential equation (in s) for the map

$$\mathcal{R}_{s,t} := \frac{\partial \tilde{m}_{s,t}}{\partial t} + \tilde{m}_{s,t} \circ V(\beta_{s,t}). \tag{42}$$

Since $\tilde{m}_{0,t} = m_0$ and $\beta_{0,t} = 0$, one has the initial condition $\mathcal{R}_{0,t} = 0$. Hence, by uniqueness of solutions of ordinary differential equations it follows that $\mathcal{R}_{s,t} = 0$ for all s. In particular, $\mathcal{R}_{1,t} = 0$ gives the desired equation

$$\frac{\partial m_t}{\partial t} + m_t \circ V(\beta_t) = 0$$

where $\beta_t = \beta_{1,t}$ has the scaling property $\beta_t = t^{-1}r_t^*\beta$. $\square$

3.3. Analyticity. As pointed out in Remark 2.3, it is natural to impose the additional condition on the KV solution β, that its components β^1, β^2 are given by (universal) analytic Lie series. We will now verify that the solution β constructed in the last section does indeed have this property.

We introduce the following terminology. Let $\mathfrak{l} = \bigoplus_{n=1}^\infty \mathfrak{l}^n$ be the free Lie algebra on generators x, y, and $\hat{\mathfrak{l}}$ its degree completion. An element

$l \in \mathfrak{l}^n$ will be called a *Lie word of length n* if it is obtained by bracketing n elements $x_{i_1}, \ldots, x_{i_n} \in \{x, y\}$, in any order. For instance, $[y, [[x, y], x]]$ is a Lie word of length 4.

Definition 3.4. Given a Lie series $l = \sum_{n=1}^{\infty} l_n \in \hat{\mathfrak{l}}$, let $C_n(l)$ denote the smallest possible $\sum_i |c_{n,i}|$ for presentations

$$l_n = \sum_i c_{n,i} l_{n,i} \tag{43}$$

where $l_{n,i}$ are Lie words of length n. The Lie series l is called *strongly convergent* if $C_n(l) \leq D^n$ for some $D > 0$.

The Campbell-Hausdorff series is an example of a strongly convergent Lie series. For any finite-dimensional Lie algebra $\mathfrak{g}$, a strongly convergent Lie series l defines an analytic function $l(x, y)$ with values in $\mathfrak{g}$ on some neighborhood of the origin in $\mathfrak{g} \times \mathfrak{g}$. Strongly convergent Lie series form a Lie subalgebra of $\hat{\mathfrak{l}}$.

Motivated by (37), we introduce a new bracket on $\hat{\mathfrak{l}} \times \hat{\mathfrak{l}}$, as follows. For any $\beta = (\beta^1, \beta^2)$, let L_β denote the derivation of $\hat{\mathfrak{l}}$ given on generators by $L_\beta x = [\beta^1, x]$, $L_\beta y = [\beta^2, y]$. If $\gamma = (\gamma^1, \gamma^2)$ is a pair of Lie series, we set $L_\beta \gamma = (L_\beta \gamma^1, L_\beta \gamma^2)$. Now put

$$[\beta, \gamma] = L_\beta \gamma - L_\gamma \beta + [\beta, \gamma]_0. \tag{44}$$

where $[\cdot, \cdot]_0$ denotes the original (componentwise) bracket. The Jacobi identity for the new bracket is a simple corollary of the derivation property for L_β.

Let us write $\mathrm{ad}_\beta = [\beta, \cdot]$, and denote by $C_n(\beta)$ the maximum of $C_n(\beta^1)$ and $C_n(\beta^2)$.

Lemma 3.5. *Suppose $\beta, \gamma \in \hat{\mathfrak{l}} \times \hat{\mathfrak{l}}$ with $C_n(\beta), C_n(\gamma) \leq D^n$ for some $D > 0$. Then $C_n([\beta, \gamma]) \leq n^2 D^n$. More generally, if $C_n(\beta_i), C_n(\gamma) \leq D^n$ then*

$$C_n(\mathrm{ad}_{\beta_k} \cdots \mathrm{ad}_{\beta_1} \gamma) \leq \frac{(2n^2)^k}{(2k-1)!!} D^n. \tag{45}$$

Here $(2k-1)!! = (2k-1)(2k-3) \cdots 1$ for all $k > 0$.

Proof. Note that if β, γ are homogeneous of degree n_1, n_2, then $[\beta, \gamma]$ is homogeneous of degree $n = n_1 + n_2$, and

$$C_n([\beta, \gamma]) \leq (n+1) C_{n_1}(\beta) C_{n_2}(\gamma).$$

This follows since $C_n(L_\beta \gamma) \leq n_2 C_{n_1}(\beta) C_{n_2}(\gamma)$, and similarly $C_n(L_\gamma \beta) \leq n_1 C_{n_1}(\beta) C_{n_2}(\gamma)$, while $C_n([\beta, \gamma]_0) \leq C_{n_1}(\beta) C_{n_2}(\gamma)$. We now obtain (45) by induction on k. The case $k = 0$ is trivial, and for $k > 0$ we find, using the induction hypothesis for $k - 1$,

$$\begin{aligned} C_n(\mathrm{ad}_{\beta_k} \cdots \mathrm{ad}_{\beta_1} \gamma) &\leq (n+1) \sum_{j=1}^{n-1} C_{n-j}(\beta_k) C_j(\mathrm{ad}_{\beta_{k-1}} \cdots \mathrm{ad}_{\beta_1} \gamma) \\ &\leq (n+1) \frac{1}{(2k-3)!!} \sum_{j=1}^{n-1} (2j^2)^{(k-1)} D^n \\ &\leq 2^{k-1}(n+1) \frac{n^{2k-1}}{(2k-1)!!} D^n \leq \frac{2^k n^{2k}}{(2k-1)!!} D^n. \end{aligned}$$

In the last line, we used $\sum_{j=1}^{n-1} j^{2k-2} \leq \int_1^n j^{2k-2}\, dj$. □

We will need one more remark. Suppose $l = \sum_{n=1}^{\infty} l_n \in \hat{\mathfrak{l}}$ is a strongly convergent Lie series. Then $l_t = r_t^* l = \sum_{n=1}^{\infty} t^n l_n$ is strongly convergent for all $0 \leq t \leq 1$, and since $t^n \leq 1$ one has the uniform estimate $C_n(l_t) \leq C_n(l) \leq D^n$. Similarly, $\dot{l}_t = \frac{\partial l_t}{\partial t}$ is strongly convergent, with a uniform estimate on $C_n(\dot{l}_t)$.

Proposition 3.6. *Let $\gamma_s \in \hat{\mathfrak{l}} \times \hat{\mathfrak{l}}$, with coefficients depending continuously on $s \in [0, 1]$, and such that $C_n(\gamma_s) \leq D^n$. Let $\gamma_{s,t} = r_t^* \gamma_s$. Then the formal sum*

$$\beta_{s,t} = \sum_{k=0}^{\infty} \int_{0 \leq s_0 \leq s_1 \cdots \leq s_k \leq s} ds_0 \ldots ds_k \left(\mathrm{ad}_{\gamma_{s_k,t}} \cdots \mathrm{ad}_{\gamma_{s_1,t}} \frac{\partial \gamma_{s_0,t}}{\partial t} \right) \tag{46}$$

defines a strongly convergent Lie series for $(s, t) \in [0, 1]^2$, with $\beta_{0,t} = 0$. Moreover, $\beta_{s,t} = t^{-1} r_t^* \beta_s$ and the following zero curvature equation holds true,

$$\partial_s \beta_{s,t} - \partial_t \gamma_{s,t} + [\beta_{s,t}, \gamma_{s,t}] = 0. \tag{47}$$

Proof. Note first of all that the right hand side of (46) defines a Lie series, since only indices $k \leq n$ contribute to the term of degree n in $\beta_{s,t}$. Also, $\beta_{0,t} = 0$ since each term on the right hand side of (46) vanishes at $s = 0$. The flatness condition (47) is satisfied as an equality of formal Lie series. To show that the series $\beta_{s,t}$ is strongly convergent, choose $D > 0$ with $C_n(\gamma_{s,t}),\ C_n(\dot{\gamma}_{s,t}) \leq D^n$. Using the estimate (45), we obtain

$$\begin{aligned} C_n(\beta_{s,t}) &\leq \sum_{k=0}^{\infty} \int_{0 \leq s_0 \leq s_1 \cdots \leq s_k \leq s} ds_0 \ldots ds_k\, C_n\!\left(\mathrm{ad}_{\gamma_{s_k,t}} \cdots \mathrm{ad}_{\gamma_{s_1,t}} \frac{\partial \gamma_{s_0,t}}{\partial t} \right) \\ &\leq \sum_{k=0}^{\infty} \frac{s^k}{k!} \frac{2^k n^{2k}}{(2k-1)!!} D^n \leq \sum_{k=0}^{\infty} \frac{2^{2k} n^{2k}}{(2k)!} D^n \leq e^{2n} D^n. \end{aligned}$$

The scaling property $\beta_{s,t} = t^{-1} r_t^* \beta_s$ holds since it is satisfied by each term in the sum (46). □

We now apply these results to the Torossian function, $\gamma_s = \tilde{\beta}_s$. The diagrammatic technique from Torossian's paper [14] gives $\tilde{\beta}_s$ directly as a universal Lie series. (The family of maps $\tilde{\beta}_s$ is however not canonical, since it depends on the choice of a suitable path in $\overline{C}^+_{2,0}$, the compactified configuration space of two points in the upper half plane.) The estimates of [4] (see also the Remark after Theorem 4.2 in [14]) show that this Lie series is strongly convergent (with coefficients depending on the parameter s), and with $C(\tilde{\beta}_s)$ uniformly bounded for $s \in [0, 1]$. Define $\beta_{s,t}$ by the Lie series (46), and identify $\beta_{s,t}$ and $\tilde{\beta}_{s,t}$ with the corresponding analytic functions on $\mathfrak{g} \times \mathfrak{g}$ (defined near the origin). By the uniqueness property for solutions of the zero curvature equation (cf. Lemma 3.2), the function $\beta_{1,t} = t^{-1} r_t^* \beta$ coincides with the KV solution from Theorem 3.3. We have hence shown that our KV solution is given by a universal, strongly convergent Lie series.

Remark 3.7. One can replace any solution $\beta = (\beta^1, \beta^2)$ of the KV conjecture with a *symmetric* solution $\beta_{\mathrm{sym}} = (\beta^1_{\mathrm{sym}}, \beta^2_{\mathrm{sym}})$ where

$$\beta^1_{\mathrm{sym}}(x, y) = \tfrac{1}{2}\big(\beta^1(x, y) + \beta^2(-y, -x)\big),$$

$$\beta^2_{\mathrm{sym}}(x, y) = \tfrac{1}{2}\big(\beta^2(x, y) + \beta^1(-y, -x)\big).$$

This solution has the additional property $\beta^1_{\mathrm{sym}}(x, y) = \beta^2_{\mathrm{sym}}(-y, -x)$.

While Theorem 3.3 settles the KV conjecture in the form stated in [8, p. 250], there remain some interesting open problems in this context.

(a) The algebraic version III of the KV conjecture can also be considered for Lie algebras over $\mathbb{Q}$. We do not know, however, whether the coefficients of our solution $\beta(x, y)$ are rational, for a suitable choice of Torossian's function $\tilde{\beta}_s(x, y)$.
(b) In a recent paper, Alekseev and Petracci [2] proved that for universal symmetric solutions β of the KV problem (for arbitrary finite-dimensional Lie algebras over $\mathbb{R}$), the linear terms in x or in y (i.e. the terms $(\partial_x \beta)(0, y)$ and $(\partial_y \beta)(x, 0)$) are uniquely determined. The linear terms in Kashiwara and Vergne's solution for the case of solvable Lie algebras [8, Equation (5.2)] are as prescribed by this result. It is unknown whether their formula might in fact give a solution for all Lie algebras.

4. Lie algebra cohomology and the KV conjecture

In this section, we will give yet another version of the Kashiwara–Vergne conjecture. This reformulation depends on a general fact in Lie algebra cohomology, stated in Proposition 4.1 below.

4.1. Lie algebra cohomology. For any $x \in \mathfrak{g}$, we denote by $\iota^\wedge(x)$ the derivation of $\wedge\mathfrak{g}^*$ given by contraction. For $\mu \in \mathfrak{g}^*$ we denote by $\epsilon(\mu)$ the operator of exterior multiplication on $\wedge\mathfrak{g}^*$.

For any $\mathfrak{g}$-module $\mathcal{M}$, let $L^{\mathcal{M}}(x) \in \mathrm{End}(\mathcal{M})$ denote the action of $x \in \mathfrak{g}$. The Lie algebra cohomology $H^\bullet(\mathfrak{g}, \mathcal{M})$ of $\mathfrak{g}$ with coefficients in $\mathcal{M}$ is the cohomology of the Chevalley-Eilenberg complex

$$C^k(\mathfrak{g}, \mathcal{M}) := \mathcal{M} \otimes \wedge^k\mathfrak{g}^*, \;\; \mathrm{d} = 1 \otimes \mathrm{d}^\wedge + \sum_a L^{\mathcal{M}}(e_a) \otimes \epsilon(e^a). \tag{48}$$

Here $e_a \in \mathfrak{g}$ and $e^a \in \mathfrak{g}^*$ are dual bases, and $\mathrm{d}^\wedge = \frac{1}{2}\sum_a \epsilon(e^a)L^\wedge(e_a)$ is the Lie algebra differential on $\wedge\mathfrak{g}^*$. Suppose $\mathfrak{k}$ is a another Lie algebra, with basis f_i, and that $\mathcal{N}$ is a $\mathfrak{k}$-module. Consider the linear map

$$V\colon \mathrm{End}(\mathcal{N}) \otimes \mathfrak{k} \to \mathrm{End}(\mathcal{N}), \;\; R = \sum_i R^i \otimes f_i \mapsto \sum_i R^i \circ L^{\mathcal{N}}(f_i). \tag{49}$$

If $\psi\colon \mathfrak{g} \to \mathfrak{k}$ is a Lie algebra homomorphism and $R \in \mathrm{End}(\mathcal{N}) \otimes \mathfrak{k}$ is $\mathfrak{g}$-invariant, then $V(R) \in \mathrm{End}(\mathcal{N})$ is $\mathfrak{g}$-equivariant. Hence it defines a chain map, $V(R) \otimes \psi^*\colon C(\mathfrak{k}, \mathcal{N}) \to C(\mathfrak{g}, \mathcal{N})$. We will need the following fact:

Proposition 4.1. *Let $\mathcal{N}$ be a $\mathfrak{k}$-module, and $\psi\colon \mathfrak{g} \to \mathfrak{k}$ a Lie algebra homomorphism. Suppose $R \in \mathrm{End}(\mathcal{N}) \otimes \mathfrak{k}$ is $\mathfrak{g}$-invariant. Then the chain map $V(R) \otimes \psi^*$ is homotopic to the trivial map. In fact,*

$$V(R) \otimes \psi^* = (1 \otimes \psi^*) \circ [d, \iota(R)] \tag{50}$$

with $\iota(R) = \sum_i R^i \otimes \iota(f_i)$.

Proof. We have to show that $1 \otimes \psi^*\colon \mathcal{N} \otimes \wedge\mathfrak{k}^* \to \mathcal{N} \otimes \wedge\mathfrak{g}^*$ annihilates the expression, $[\mathrm{d}, \iota(R)] - V(R) \otimes 1$. Let $f^i \in \mathfrak{k}^*$ be the dual basis to $f_i \in \mathfrak{k}$. We compute,

$$\begin{aligned}
&[\mathrm{d}, \iota(R)] - V(R) \otimes 1 \\
&\qquad = \Big[1 \otimes \mathrm{d}^\wedge + \sum_j L^{\mathcal{N}}(f_j) \otimes \epsilon(f^j), \sum_i R^i \otimes \iota(f_i)\Big] - V(R) \otimes 1 \\
&\qquad = \sum_i R^i \otimes L^\wedge(f_i) + \sum_{ij} L^{\mathcal{N}}(f_j)(R^i) \otimes \big(\epsilon(f^j) \circ \iota(f_i)\big).
\end{aligned}$$

Since $\sum_j (\psi^* f^j) \otimes f_j = \sum_a e^a \otimes \psi(e_a)$, we have the following equalities of maps $\wedge\mathfrak{k}^* \to \wedge\mathfrak{g}^*$,

$$\begin{aligned}
\psi^* \circ L^\wedge(f_i) &= \sum_j \psi^* \circ \epsilon(f^j) \circ \iota([f_j, f_i]_{\mathfrak{k}}) \\
&= \sum_j \epsilon(\psi^* f^j) \circ \psi^* \circ \iota([f_j, f_i]_{\mathfrak{k}}) \\
&= \sum_a \epsilon(e^a) \circ \psi^* \circ \iota([\psi(e_a), f_i]_{\mathfrak{k}}).
\end{aligned}$$

One the other hand, using the $\mathfrak{g}$-equivariance of R,

$$\begin{aligned}(1 \otimes \psi^*) \circ \Big(\sum_{ij} L^{\mathcal{N}}(f_j)(R^i) \otimes \epsilon(f^j) \circ \iota(f_i)\Big) \\ &= \sum_{ia} L^{\mathcal{N}}(\psi(e_a))(R^i) \otimes \epsilon(e^a) \circ \psi^* \circ \iota(f_i) \\ &= -\sum_{ia} R^i \otimes \big(\epsilon(e^a) \circ \psi^* \circ \iota([\psi(e_a), f_i]_{\mathfrak{k}})\big).\end{aligned}$$

□

4.2. Reformulation of the KV conjecture. We will apply this proposition to the diagonal embedding $\psi \colon \mathfrak{g} \to \mathfrak{g}^2$. The map $\psi^* \colon \wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g}^* = \wedge(\mathfrak{g}^* \oplus \mathfrak{g}^*) \to \wedge \mathfrak{g}^*$ is just the product map in the exterior algebra. View $\mathcal{N} = \mathcal{D}'_{\mathrm{comp}}(O^2)$ as a $\mathfrak{g}^2$-module, and $\mathcal{M} = \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g})$ as a $\mathfrak{g}$-module. We obtain a family of chain maps,

$$M_t = m_t \otimes \psi^* \colon C\big(\mathfrak{g}^2, \mathcal{D}'_{\mathrm{comp}}(O^2)\big) \to C\big(\mathfrak{g}, \mathcal{D}'_{\mathrm{comp}}(\mathfrak{g})\big). \tag{51}$$

Version I of the Kashiwara–Vergne conjecture says

$$\frac{\partial m_t}{\partial t} = -m_t \circ V(\beta_t), \tag{52}$$

where $\beta_t \in C^\infty(M, \mathfrak{k})$. (The components β_t^i are identified with the corresponding operators of multiplication on $\mathcal{D}'_{\mathrm{comp}}(O^2)$.) Tensoring with $\psi^* \colon \wedge^2 \mathfrak{g}^* \to \wedge \mathfrak{g}^*$, and using Proposition 4.1, we obtain the following equivalent reformulation of the KV conjecture:

Kashiwara–Vergne conjecture, version IV: *Taking O smaller if necessary, there exists a $\mathfrak{g}$-equivariant function $\beta \colon O^2 \to \mathfrak{g}^2$, with $\beta(0,0) = 0$, such that $\beta_t = t^{-1} r_t^* \beta$ satisfies*

$$\frac{\partial M_t}{\partial t} = -M_t \circ [d, \iota(\beta_t)]. \tag{53}$$

The algebra isomorphism $H(\mathfrak{g}, S\mathfrak{g}) \to H(\mathfrak{g}, U\mathfrak{g})$ is an immediate consequence of this conjecture. More generally, one obtains a similar statement for convolution of germs of distributions. Let $A_1, A_2 \subset \mathfrak{g}$ be closed cones with $A_1 \cap -A_2 = \{0\}$, and consider the diagram

$$\begin{array}{ccc} C(\mathfrak{g}, \mathfrak{D}'(\mathfrak{g}, A_1)) \times C(\mathfrak{g}, \mathfrak{D}'(\mathfrak{g}, A_1)) & \longrightarrow & C(\mathfrak{g}, \mathfrak{D}'(\mathfrak{g}, A_1 + A_2)) \\ \downarrow & & \downarrow \\ C(\mathfrak{g}, \mathfrak{D}'(G, A_1)) \times C(\mathfrak{g}, \mathfrak{D}'(G, A_1)) & \longrightarrow & C(\mathfrak{g}, \mathfrak{D}'(G, A_1 + A_2)) \end{array}$$

where the horizontal maps are product maps, and the vertical maps are induced by the Duflo maps. Version IV of the Kashiwara–Vergne conjecture implies that this diagram commutes up to a chain homotopy. In particular, passing to cohomology one obtains a commutative diagram

$$\begin{array}{ccc} H(\mathfrak{g}, \mathfrak{D}'(\mathfrak{g}, A_1)) \otimes H(\mathfrak{g}, \mathfrak{D}'(\mathfrak{g}, A_2)) & \longrightarrow & H(\mathfrak{g}, \mathfrak{D}'(\mathfrak{g}, A_1 + A_2)) \\ \downarrow & & \downarrow \\ H(\mathfrak{g}, \mathfrak{D}'(G, A_1)) \otimes H(\mathfrak{g}, \mathfrak{D}'(G, A_2)) & \longrightarrow & H(\mathfrak{g}, \mathfrak{D}'(G, A_1 + A_2)) \end{array}$$

4.3. Higher homotopies. Version IV of the KV conjecture shows that each map $\mathrm{Duf}_t \otimes \mathrm{id} : S(\mathfrak{g}) \otimes \wedge\mathfrak{g}^* \to U(\mathfrak{g}) \otimes \wedge\mathfrak{g}^*$ is an algebra isomorphism up to homotopy. Solutions of the KV problem provide infinitesimal homotopies for family of products M_t. One can ask whether in fact $\mathrm{Duf}_t \otimes \mathrm{id}$ extend to an A_∞-morphisms (for a definition see [10]) and to construct infinitesimal homotopies for its higher components. Indeed, let

$$M_t^{(3)} = M_t \circ (M_t \otimes 1) \colon C\big(\mathfrak{g}^3, \mathcal{D}'_{\mathrm{comp}}(O^3)\big) \to C(\mathfrak{g}, \mathcal{D}_{\mathrm{comp}}(\mathfrak{g})) \tag{54}$$

denote the triple product map (defined for O sufficiently small). Let β_t denote a solution of the KV-problem. Taking the t-derivative of $M_t^{(3)} = M_t \circ (M_t \otimes 1)$, we obtain $\dot{M}_t^{(3)} = -[\mathrm{d}, h_t^{(3)}]$ where $h_t^{(3)}$ is the homotopy operator

$$h_t = M_t^{(3)} \circ (\iota(\beta_t) \otimes 1) + M_t \circ \iota(\beta_t) \circ (M_t \otimes 1). \tag{55}$$

On the other hand, starting with $M_t^{(3)} = M_t \circ (1 \otimes M_t)$, we get another homotopy operator

$$\tilde{h}_t = M_t^{(3)} \circ (1 \otimes \iota(\beta_t)) + M_t \circ \iota(\beta_t) \circ (1 \otimes M_t). \tag{56}$$

The differences $\tilde{h}_t - h_t$ are cochain maps of degree -1. In the spirit of version *IV* of the KV conjecture, one can formulate the following new conjecture: *For $O \subset \mathfrak{g}$ a sufficiently small neighborhood of the origin, there exists a $\mathfrak{g}$-equivariant map $\beta^{(3)} \colon O^3 \to \wedge^2\mathfrak{g}^3$, with $\beta^{(3)}(0) = 0$, such that $\beta_t^{(3)} = t^{-1} r_t^* \beta^{(3)}$ satisfies*

$$\tilde{h}_t - h_t = -M_t^{(3)} \circ \big[d, \iota\big(\beta_t^{(3)}\big)\big]. \tag{57}$$

In a similar fashion, one can conjecture the existence of higher homotopies given by equivariant functions $\beta^{(n)} \colon O^n \to \wedge^{n-1}\mathfrak{g}^n$.

Appendix A. The zero curvature equation

In this appendix we prove Lemma 3.2, the zero curvature equation for the Lie algebra $\mathfrak{k}_M$. Let us first recall the more general zero curvature equation for vector fields on a manifold M.

Lemma A.1. *Let $Y_{s,t} \in \mathfrak{X}(M)$ a given 2-parameter family of vector fields, depending smoothly on $s, t \in [0, 1]$, and assume that $Y_{s,t}$ vanishes at some given point $x_0 \in M$ for all s, t. Then there exists an open neighborhood U of x_0, and a unique 2-parameter family of vector fields $X_{s,t}$, $s, t \in [0, 1]$, satisfying the zero curvature equation*

$$\frac{\partial X_{s,t}}{\partial s} - \frac{\partial Y_{s,t}}{\partial t} + [X_{s,t}, Y_{s,t}] = 0 \tag{58}$$

with initial condition $X_{0,t} = 0$.

Proof. Let $F_{s,t}$ denote the flow of $Y_{s,t}$, viewed as an s-dependent vector field depending on t as a parameter. In terms of the action on smooth functions f,

$$(Y_{s,t} f) \circ F_{s,t}^{-1} = \frac{\partial}{\partial s}\big(f \circ F_{s,t}^{-1}\big); \quad F_{0,t} = \mathrm{id}\,. \tag{59}$$

Since $Y_{s,t}(x_0) = 0$, there exists an open neighborhood U of x_0 such that $F_{s,t}(x)$ is defined for all $x \in U$ and $s, t \in [0, 1]$. Define a vector field $X_{s,t}$ by

$$(X_{s,t} f) \circ F_{s,t}^{-1} = \frac{\partial}{\partial t}\big(f \circ F_{s,t}^{-1}\big). \tag{60}$$

Note that $X_{0,t} = 0$ since $F_{0,t} = \mathrm{id}$. Subtracting the s-derivative of (60) from the t-derivative of (59) one finds that $X_{s,t}$ solves (58). On the other hand, any two solutions of (58) differ by a 2-parameter family of vector fields $Z_{s,t}$ with $\frac{\partial}{\partial s} Z_{s,t} + [Z_{s,t}, Y_{s,t}] = 0$, or equivalently $\frac{\partial}{\partial s}((F_{s,t}^{-1})^* Z_{s,t}) = 0$. Integrating with initial condition $Z_{0,t} = 0$ one obtains $Z_{s,t} = 0$, proving uniqueness. □

Lemma 3.2 (the zero curvature equation in $\mathfrak{k}_M$) may be viewed as a special case of Lemma A.1. Indeed, $\mathfrak{k}_M$ may be realized as a Lie algebra of vector fields on $K \times M$, where K is a Lie group with Lie algebra $\mathfrak{k}$. To see this, lift the Lie algebra $\mathfrak{k}$-action on M to an action on $K \times M$ by $\xi \mapsto \hat{X}^\xi = (\xi^L, X^\xi)$ where $\xi^L \in \mathfrak{X}(K)$ is the left-invariant vector field generated by ξ. Accordingly, the homomorphism $\mathfrak{k}_M \to \mathfrak{X}(M)$ lifts to a Lie algebra homomorphisms

$$\mathfrak{k}_M \to \mathfrak{X}(K \times M), \quad \beta \mapsto \hat{X}^\beta. \tag{61}$$

It is clear that the map (61) is injective. Its image consists of vector fields on $K \times M$ that are invariant under the K-action by left-multiplication on the first factor, and are tangent to the foliation defined by the $\mathfrak{k}$-action $\xi \mapsto \hat{X}^\xi$ on $K \times M$. (Equivalently, the flow of such vector fields is K-equivariant, and preserves the leaves of the $\mathfrak{k}$-action.) Using (61) to identify elements of $\mathfrak{k}_M$ with vector fields, Lemma 3.2 is now a direct consequence of (58).

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